Mean-Field Dynamics of the Bose-Hubbard Model in High Dimension

[1]

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11/02/2025

1 Motivations

Usual many-body $N \to \infty$ mean field:

$$H_N \coloneqq \sum_{i=1}^{N} (-\Delta_i) + \frac{1}{N} \sum_{1 \leqslant i < j \leqslant N} w(X_i - X_j)$$

Our goal

- Mean field limit as $d \to \infty$
- Describe a phase transition
- Strong particle interactions

Quantum phase transition from a superfluid to a Mott insulator in an ultracold gas of atoms

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Abstract

A quantum phase transition from a superfluid to a Mott insulating ground state was observed in a Bose–Einstein condensate stored in a three-dimensional optical lattice potential. With this experiment a new field of physics with ultracold atomic quantum gases is entered. Now interactions between atoms dominate the behavior of the many-body system, such that it cannot be described by the usual theories for weakly interacting Bose gases anymore.
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Experimental observation of the phase transition [2]

Model: interacting bosons on a lattice

- Simple mathematical description
- Great success in physics:
  Mott-insulator \ Superfluid phase transition
- Mean field justified when $d \to \infty$
  and effective in $d = 3$

PHYSICAL REVIEW B VOLUME 40, NUMBER 1 1 JULY 1989

Boson localization and the superfluid-insulator transition

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Theoretical description of the mean field theory [3]

Result: convergence of the many-body dynamics to the mean field dynamics when $d \to \infty$

2 Bose-Hubbard model

Lattice: $\Lambda := (\mathbb{Z}/L\mathbb{Z})^d$ with $d, L \in \mathbb{N}$ such that $d, L \geqslant 2$ of volume $|\Lambda| = L^d$

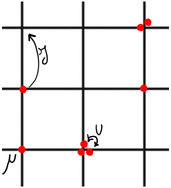
One-lattice-site Hilbert space: $\ell^2(\mathbb{C})$ of canonical basis $|n\rangle := (0, \dots, 0, \underbrace{1}_{n^{th}\text{index}}, 0, \dots), n \in \mathbb{N}$

2^{nd} **quantization:** creation and annihilation operators:

$$a\,|0\rangle := 0 \quad \forall n \in \mathbb{N}^*,\ a\,|n\rangle := \sqrt{n}\,|n-1\rangle\,, \tag{1}$$

$$\forall n \in \mathbb{N},\ a^\dagger\,|n\rangle := \sqrt{n+1}\,|n+1\rangle \tag{2}$$

$$[a, a^\dagger] = 1 \tag{CCR}$$



Particle number: $\mathcal{N} := a^\dagger a$

Fock space:

$$\ell^2(\mathbb{C})^{\otimes|\Lambda|} \cong \mathcal{F}_+\left(L^2(\Lambda, \mathbb{C})\right) := \bigoplus_{n\in\mathbb{N}} L^2(\Lambda, \mathbb{C})^{\otimes_+ n} \tag{3}$$

Bose-Hubbard hamiltonian of parameters $J, \mu, U \in \mathbb{R}$:

$$H_d := -\frac{J}{2d} \overbrace{\sum_{\substack{x,y\in\lambda \\ x\sim y}} a_x^\dagger a_y}^{\mathcal{O}(2d|\Lambda|)} + (J-\mu)\sum_{x\in\Lambda} \mathcal{N}_x + \frac{U}{2}\sum_{x\in\Lambda} \mathcal{N}_x(\mathcal{N}_x - 1) \tag{4}$$

Dynamics for $\gamma_d \in L^\infty\left(\mathbb{R}_+, \mathcal{L}^1\left(\ell^2(\mathbb{C})^{\otimes|\Lambda|}\right)\right)$:

$$i\partial_t \gamma_d(t) = [H_d, \gamma_d(t)] \tag{B-H}$$

First reduced one-lattice-site density matrix:

$$\gamma_d^{(1)} := \frac{1}{|\Lambda|}\sum_{x\in\Lambda} \mathrm{Tr}_{\Lambda\setminus\{x\}}(\gamma_d) \tag{5}$$

3 Mean field theory

Mean field hamiltonian for $\varphi \in \ell^2(\mathbb{C})$:

$$h^{\varphi} := -J\left(\overline{\alpha_\varphi} a + \alpha_\varphi a^\dagger - |\alpha_\varphi|^2\right) + (J-\mu)\mathcal{N} + \frac{U}{2}\mathcal{N}(\mathcal{N}-1) \tag{6}$$

with the order parameter

$$\alpha_\varphi := \langle \varphi | a\varphi \rangle$$

Phase transition: Decompose

$$\varphi =: \sum_{n\in\mathbb{N}} \lambda_n |n\rangle \implies \alpha_\varphi = \sum_{n\in N} \sqrt{n+1}\,\overline{\lambda_n}\lambda_{n+1}$$

- Mott Insulator (MI): $\alpha_\varphi = 0$
- Superfluid (SF): $\alpha_\varphi > 0$

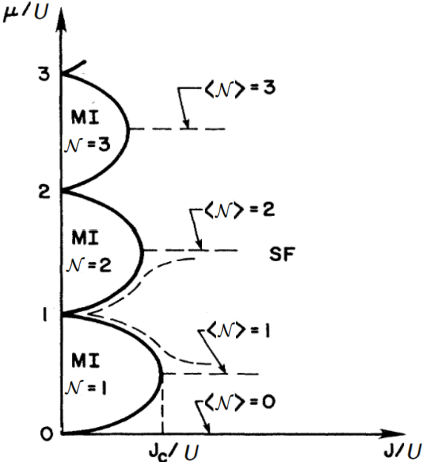
Dynamics

For $\varphi \in L^\infty\left(\mathbb{R}_+, \ell^2(\mathbb{C})\right)$,

$$i\partial_t \varphi(t) = h^{\varphi(t)}\varphi(t) \tag{mf}$$

Corresponding projection

$$p_\varphi := |\varphi\rangle\langle\varphi| \tag{7}$$



Mott insulator \ Superfluid phase diagram obtained by minimizing $\varphi \mapsto \langle\varphi|h^\varphi\varphi\rangle$[3]

4 Main result

Recap

$$\gamma_d^{(1)} := \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\mathrm{Tr}_{\Lambda\setminus\{x\}}(\gamma_d) \quad i\partial_t\gamma_d = \left[-\frac{J}{2d}\sum_{\substack{x,y\in\lambda\\ x\sim y}} a_x^\dagger a_y + (J-\mu)\sum_{x\in\Lambda}\mathcal{N}_x + \frac{U}{2}\sum_{x\in\Lambda}\mathcal{N}_x(\mathcal{N}_x-1), \gamma_d\right] \quad \text{(B-H)}$$

$$\alpha_\varphi := \langle\varphi|a\varphi\rangle \quad i\partial_t\varphi = \left(-J(\alpha_\varphi a + \overline{\alpha_\varphi}a^\dagger - |\alpha_\varphi|^2) + (J-\mu)\mathcal{N} + \frac{U}{2}\mathcal{N}(\mathcal{N}-1)\right)\varphi \quad \text{(mf)}$$

Theorem: *S.Farhat D.P S.Petrat 2025*

Assume

- γ_d solves (B-H) with $\gamma_d(0)\in\mathcal{L}^1\left(\ell^2(\mathbb{C})^{\otimes|\Lambda|}\right)$ such that $\mathrm{Tr}\,(\gamma_d(0)) = 1$
- φ solves (mf) with $\varphi(0)\in\ell^2(\mathbb{C})$ such that $\|\varphi\|_{\ell^2} = 1$
- $\exists c_1, c_2 > 0$ such that $\forall n\in\mathbb{N}$,

$$\mathrm{Tr}\,(p_\varphi(0)\mathbb{1}_{\mathcal{N}=n}) \leqslant c_1 e^{-\frac{n}{c_2}} \quad \mathrm{Tr}\left(\gamma_d^{(1)}(0)\mathbb{1}_{\mathcal{N}=n}\right) \leqslant c_1 e^{-\frac{n}{c_2}}. \tag{8}$$

Then $\exists\ C := C\,(J, c_1, c_2, \mathrm{Tr}\,(p_\varphi(0)\mathcal{N})) > 0$ such that $\forall t\in\mathbb{R}_+$,

$$\left\|\gamma_d^{(1)}(t) - p_\varphi(t)\right\|_{\mathcal{L}^1} \leqslant Ce^{Cte^{Ct}\sqrt{\ln(d)}}\left(\left\|\gamma_d^{(1)}(0) - p_\varphi(0)\right\|_{\mathcal{L}^1} + \frac{1}{d\sqrt{\ln(d)}}\right) \tag{9}$$

If $\left\|\gamma_d^{(1)}(0) - p_\varphi(0)\right\|_{\mathcal{L}^1} = \mathcal{O}\left(\frac{1}{d}\right)$, then $\forall t\in\mathbb{R}_+$,

$$\left\|\gamma_d^{(1)}(t) - p_\varphi(t)\right\|_{\mathcal{L}^1} \lesssim e^{Cte^{Ct}\sqrt{\ln(d)}-\ln(d)} \underset{d\to\infty}{\longrightarrow} 0$$

Thank you for your attention

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